Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number

Candidate Number

Pearson Edexcel Level 3 GCE

Wednesday 6 October 2021 – Afternoon

Time 2 hours

Paper reference **9MA0/01**

Mathematics

Advanced

PAPER 1: Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 15 questions in this question paper. The total mark for this paper is 100.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

P68731A

A:1/1/1/1/1/

P 6 8 7 3 1 A 0 1 5 2

Pearson

1. $$f(x) = ax^3 + 10x^2 - 3ax - 4$$

Given that $(x - 1)$ is a factor of $f(x)$, find the value of the constant a.

You must make your method clear.

(3)

If $(x-1)$ is a factor, 1 must be a root, since we use the opposite sign. So f(1) must be 0, because on a graph, this is where the line passes through the x axis.

$f(1) = 0$

$a(1)^3 + 10(1)^2 - 3a(1) - 4 = 0$

$f(1) = 1^3 a + 10 \times 1^2 - 3a \times 1 - 4 = 0$

$= a + 10 - 3a - 4 = 0$

$= 6 - 2a = 0$

$6 = 2a$

$\boxed{a = 3}$

2. Given that

$$f(x) = x^2 - 4x + 5 \qquad x \in \mathbb{R}$$

(a) express f(x) in the form $(x + a)^2 + b$ where a and b are integers to be found.

(2)

The curve with equation $y = f(x)$

- meets the y-axis at the point P
- has a minimum turning point at the point Q

(b) Write down

(i) the coordinates of P

(ii) the coordinates of Q

(2)

2. a) Complete the square to get the required form

$$f(x) = x^2 - 4x + 5$$
$$= (x-2)^2 - 4 + 5$$
$$\boxed{= (x-2)^2 + 1}$$

b) i) The +5 in the equation $f(x) = x^2 - 4x + 5$ is the y intercept.
So $y = 5$, and $x = 0$

$\boxed{P = (0, 5)}$

ii) In the form $(x+a)^2 + b$, the co-ordinates of the turning point are $(-a, b)$

$(x-2)^2 + 1 \longrightarrow \boxed{Q = (2, 1)}$

$a = -2$ $\quad b = 1$

P 6 8 7 3 1 A 0 4 5 2

3. The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_{n+1} = k - \frac{24}{u_n} \qquad u_1 = 2$$

where k is an integer.

Given that $u_1 + 2u_2 + u_3 = 0$

(a) show that

$$3k^2 - 58k + 240 = 0$$

(3)

(b) Find the value of k, giving a reason for your answer.

(2)

(c) Find the value of u_3

(1)

a) Find u_1, u_2, and u_3 using $u_{n+1} = k - \frac{24}{u_n}$

$u_1 = 2$

$u_2 = k - \frac{24}{u_1} = k - \frac{24}{2} = k - 12$

$u_3 = k - \frac{24}{u_2} = k - \frac{24}{k-12} = k - \frac{24}{k-12}$

Then find $u_1 + 2u_2 + u_3 = 0$ using the values we found

$u_1 + 2u_2 + u_3 = 0$

$2 + 2(k-12) + k - \frac{24}{k-12} = 0$

multiply all terms by $(k-12)$ to get the equation in the required form, and to get rid of the fraction.

$2(k-12) + 2(k-12)^2 + k(k-12) - 24 = 0$

$2k - 24 + 2k^2 - 48k + 288 + k^2 - 12k - 24 = 0$

$\boxed{3k^2 - 58k + 240 = 0}$

b) To find k, solve $3k^2 - 58k + 240 = 0$ by factorising the equation

$(3k-40)(k-6) = 0$

not an integer $k = \frac{40}{3}$ $k = 6$

$\boxed{k = 6 \text{, because } k \text{ must be an integer}}$

Question 3 continued

c) From part a), $u_3 = k - \frac{24}{k-12}$, and $k = 6$

$$u_3 = 6 - \frac{24}{k-6}$$

$$= 6 - \frac{24}{6-12}$$

$$= 6 + 4$$

$$\boxed{u_3 = 10}$$

(Total for Question 3 is 6 marks)

4. The curve with equation $y = f(x)$ where

$$f(x) = x^2 + \ln(2x^2 - 4x + 5)$$

has a single turning point at $x = \alpha$

(a) Show that α is a solution of the equation

$$2x^3 - 4x^2 + 7x - 2 = 0$$

(4)

The iterative formula

$$x_{n+1} = \frac{1}{7}(2 + 4x_n{}^2 - 2x_n{}^3)$$

is used to find an approximate value for α.

Starting with $x_1 = 0.3$

(b) calculate, giving each answer to 4 decimal places,

(i) the value of x_2

(ii) the value of x_4

(3)

Using a suitable interval and a suitable function that should be stated,

(c) show that α is 0.341 to 3 decimal places.

(2)

4a) At a turning point, the gradient equals 0, so $f'(x) = 0$

Find $f'(x)$ $\quad f'(x) = 2x + \dfrac{4x-4}{2x^2-4x+5}$

$$\boxed{\frac{d}{dx}\ln[f(x)] = \frac{f'(x)}{f(x)}}$$

$f'(x) = 0$ $\quad 0 = 2x + \dfrac{4x-4}{2x^2-4x+5}$

multiply all terms by $(2x^2 - 4x + 5)$

rearrange to get required form

$0 = 2x(2x^2 - 4x + 5) + 4x - 4$

expand and simplify

$0 = 4x^3 - 8x^2 + 10x + 4x - 4$

divide by 2

$$\boxed{0 = 2x^3 - 4x^2 + 7x - 4}$$

b) i) Using your calculator, sub in $x_1 = 0.3$ into $x_{n+1} = \frac{1}{7}(2 + 4x_n^2 - 2x_n^3)$ this will get x_2

$$x_2 = \frac{1}{7}\left(2 + 4(0.3^2) - 2(0.3^3)\right)$$
$$= \frac{1}{7}(2 + 0.36 - 0.054)$$
$$\boxed{x_2 = 0.3294}$$

ii) Repeat bi) by subbing x_2 into $\frac{1}{7}(2 + 4x_n^2 - 2x_n^3)$ to get x_3, then sub in x_3 to get x_4

$$x_3 = \frac{1}{7}\left(2 + 4(0.3294^2) - 2(0.3294^3)\right)$$
$$= 0.3375$$

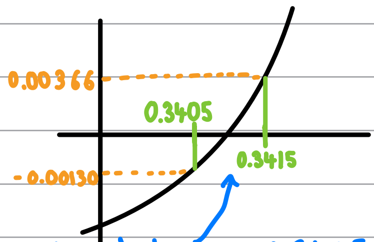
save x_2 in your calculator to increase the accuracy of x_3 and x_4.

$$x_4 = \frac{1}{7}\left(2 + 4(0.3375^2) - 2(0.3375^3)\right)$$
$$\boxed{x_4 = 0.3398}$$

c) To show that $\alpha = 0.341$ to 3dp, sub in 0.3415 and 0.3405 into $2x^3 - 4x^2 + 7x - 4$

$$f(0.3415) = 0.00366$$
$$f(0.3405) = -0.00130$$

there is a sign change between $f(0.3405)$ and $f(3415)$



somewhere between 0.3405 and 0.3415, the root is 0.

$\boxed{\text{so } \alpha = 0.341 \text{ to 3dp because there is a change of sign between } f(0.3405) \text{ and } f(0.3415)}$

5. **In this question you should show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A company made a profit of £20 000 in its first year of trading, Year 1

A model for future trading predicts that the yearly profit will increase by 8% each year, so that the yearly profits will form a geometric sequence.

According to the model,

(a) show that the profit for Year 3 will be £23 328 **(1)**

(b) find the first year when the yearly profit will exceed £65 000 **(3)**

(c) find the total profit for the first 20 years of trading, giving your answer to the nearest £1000 **(2)**

a) Year 1 profit = 20,000
Year 2 profit = 20,000 × 1.08 = 21,600
Year 3 profit = 21,600 × 1.08 = 23328

geometric sequence: $a_n = a_1(r)^{n-1}$

n = term position (3)

a_n = nth term (3rd term)
first term (20,000)
r = common ratio (1.08)

$\boxed{\text{Year 3 profit} = 20{,}000 \times 1.08^2 = 23328}$

b) Use the formula $a_n = a_1(r)^{n-1}$

$a_n = 65{,}000$
$a_1 = 20{,}000$
$r = 1.08$
n = year (what we need to find)

< symbol because we are finding when it first exceeds 65,000

$65{,}000 < 20{,}000 \times 1.08^{n-1}$
$3.25 < 1.08^{n-1}$
$\log_{1.08} 3.25 < n - 1$
$15.31 < n - 1$
$16.31 < n$

n must be greater than 16.31, so in the 17th year the profit exceeds £65,000

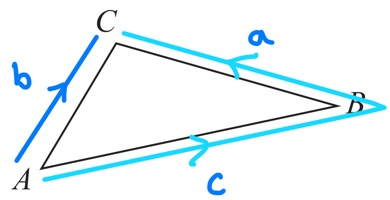
$\boxed{\text{Year 17}}$

c) Use the formula $S_n = \dfrac{a(1-r^n)}{1-r}$ in formula booklet p5

$n = 20,\ a = 20{,}000,\ r = 1.08$

$S_{20} = \dfrac{20{,}000(1-1.08^{20})}{1-1.08} = \boxed{£915{,}000}$

6.



$\vec{AB} + \vec{AC} = \vec{AC}$

Figure 1

Figure 1 shows a sketch of triangle *ABC*.

Given that

- $\overrightarrow{AB} = -3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$
- $\overrightarrow{BC} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$

(a) find $\overrightarrow{AC}$

(2)

(b) show that $\cos ABC = \dfrac{9}{10}$

(3)

a) $\vec{AC} = \vec{AB} + \vec{BC}$ so just add the two vectors together

$\vec{AB} + \vec{BC} = (-3i - 4j - 5k) + (i + j + 4k)$
$= -2i - 3j - k$

$\boxed{\vec{AC} = -2i - 3j - k}$

$\cos B = \cos ABC$

b) We can use the cos rule $\boxed{b^2 = a^2 + c^2 - 2ac\cos B}$ to find $\cos B$

Use pythagoras to find lengths a, b, c

$a = |\vec{BC}| = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18}$
$b = |\vec{AC}| = \sqrt{(-2)^2 + (-3)^2 + (-1)^2} = \sqrt{14}$
$c = |\vec{AB}| = \sqrt{(-3)^2 + (-4)^2 + (-5)^2} = \sqrt{50}$

rearrange for cos B $\quad \cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

sub in a, b, c $\quad \cos B = \dfrac{\sqrt{18}^2 + \sqrt{50}^2 - \sqrt{14}^2}{2 \times \sqrt{18} \times \sqrt{50}}$

$\cos B = \dfrac{18 + 50 - 14}{2 \times \sqrt{18} \times \sqrt{50}}$

$\cos B = \dfrac{54}{60} = \dfrac{9}{10}$

$\boxed{\cos B = \dfrac{9}{10}}$

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

P 6 8 7 3 1 A 0 1 4 5 2

7. The circle C has equation

$$x^2 + y^2 - 10x + 4y + 11 = 0$$

(a) Find

(i) the coordinates of the centre of C,

(ii) the exact radius of C, giving your answer as a simplified surd.

(4)

The line l has equation $y = 3x + k$ where k is a constant.

Given that l is a tangent to C,

(b) find the possible values of k, giving your answers as simplified surds.

(5)

a) i) To find the centre, rearrange the equation into the form $(x-a)^2 + (y-b)^2 = r^2$. Do this by completing the square of both x and y. The centre is (a,b)

$x^2 - 10x + y^2 + 4y + 11 = 0$

$(x-5)^2 - 25 + (y+2)^2 - 4 + 11 = 0$

$(x-5)^2 + (y+2)^2 - 18 = 0$

$(x-5)^2 + (y+2)^2 = 18$

$\boxed{\text{centre} = (5,-2)}$

ii) We have found the new equation, and so we have already found $r^2 = 18$

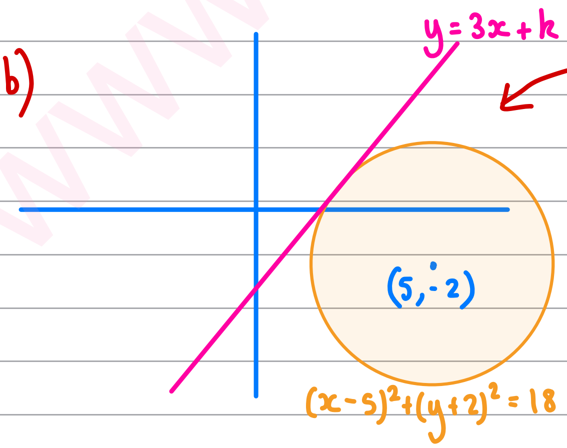
$r^2 = 18$

$r = \sqrt{18}$

$r = \sqrt{9}\sqrt{2}$

$\boxed{r = 3\sqrt{2}}$

b)



If l is a tangent, line l intersects the circle at only one point

So, we need to solve $y = 3x + k$ and $(x-5)^2 + (y+2)^2 = 18$ simultaneously to find the co-ordinates of its intersection

Question 7 continued

Sub $y = 3x + k$ into $(x-5)^2 + (y+2)^2 = 18$ to eliminate y

$(x-5)^2 + (3x+k+2)^2 = 18$

$x^2 - 10x + 25 + 9x^2 + 6xk + 12x + k^2 + 4k + 4 = 18$

$10x^2 + x(2+6k) + (k^2 + 4k + 11) = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$b^2 - 4ac < 0 \rightarrow$ no real solutions

$b^2 - 4ac = 0 \rightarrow$ one solution

$b^2 - 4ac > 0 \rightarrow$ two solutions

We need to find $b^2 - 4ac = 0$, because a tangent only has 1 real solution

$b^2 - 4ac$

$= (2+6k)^2 - 4 \times 10 \times (k^2 + 4k + 11) = 0$

$= 36k^2 + 24k + 4 - 40(k^2 + 4k + 11) = 0$

$= 36k^2 + 24k + 4 - 40k^2 - 160k - 440 = 0$

$= -4k^2 - 136k - 436 = 0$

$= 4k^2 + 136k + 436 = 0$ ← now solve again using the quadratic formula to find k

$k = \dfrac{-136 \pm \sqrt{136^2 - (4 \times 4 \times 436)}}{2 \times 4}$

$= \dfrac{-136 \pm \sqrt{11520}}{8}$

$\boxed{k = -17 \pm 6\sqrt{5}}$

P 6 8 7 3 1 A 0 1 9 5 2

8. A scientist is studying the growth of two different populations of bacteria.

The number of bacteria, N, in the **first** population is modelled by the equation

$$N = Ae^{kt} \qquad t \geqslant 0$$

where A and k are positive constants and t is the time in hours from the start of the study.

Given that

- there were 1000 bacteria in this population at the start of the study
- it took exactly 5 hours from the start of the study for this population to double

(a) find a complete equation for the model.

(4)

(b) Hence find the rate of increase in the number of bacteria in this population exactly 8 hours from the start of the study. Give your answer to 2 significant figures.

(2)

The number of bacteria, M, in the **second** population is modelled by the equation

$$M = 500e^{1.4kt} \qquad t \geqslant 0$$

where k has the value found in part (a) and t is the time in hours from the start of the study.

Given that T hours after the start of the study, the number of bacteria in the two different populations was the same,

(c) find the value of T.

(3)

a) $N = Ae^{kt}$

when $t=0$ (the start of the study), $A = N$

$\boxed{A = 1000}$

when $t = 5$, $N = 2000$ (because the population has doubled)

$2000 = 1000e^{5k}$ solve to find k

$2 = e^{5k}$

$\ln 2 = \ln e^{5k}$ $\boxed{\ln e^{f(k)} = f(k)}$

$5k = \ln 2$

$k = \ln 2/5$ now put together to get the full equation

$\boxed{N = 1000e^{(\frac{1}{5}\ln 2)t}}$

Question 8 continued

b) to find the rate of increase, differentiate N with respect to t

$$\frac{dN}{dt} = \tfrac{1}{5}\ln 2 \times 1000\, e^{(\frac{1}{5}\ln 2)t} \quad \longleftarrow \quad \boxed{\frac{d}{dx} e^{f(x)} = f'(x)e^{f(x)}}$$

$$= 200\ln 2\, e^{(\frac{1}{5}\ln 2)t}$$

sub in $t=8$ to find the rate of increase 8 hours after the start of the study

$$@t=8 \quad \frac{dN}{dt} = 200\ln 2\, e^{(\frac{1}{5}\ln 2)\times 8}$$

$$= 420.246$$

$$\boxed{= 420 \text{ to } 2sf}$$

c) If the number of bacteria in the first population = second population, $M = N$

The number of bacteria is the same at the same time, so t is equal in both populations

$$1000e^{(\frac{1}{5}\ln 2)T} = 500e^{1.4kT}$$

$k = \frac{1}{5}\ln 2$

$1.4k = \frac{7}{25}\ln 2$

$$1000e^{(\frac{1}{5}\ln 2)T} = 500e^{(\frac{7}{25}\ln 2)T}$$

solve to find T

$$2e^{(\frac{1}{5}\ln 2)T} = e^{(\frac{7}{25}\ln 2)T}$$

$$2 = \frac{e^{(\frac{7}{25}\ln 2)T}}{e^{(\frac{1}{5}\ln 2)T}} \quad \boxed{\frac{e^x}{e^y} = e^{x-y}}$$

$$2 = e^{(\frac{7}{25}\ln 2 - \frac{1}{5}\ln 2)T}$$

$$2 = e^{(\frac{2}{25}\ln 2)T}$$

take the natural log of both sides to get rid of e.

$$\ln 2 = \ln e^{(\frac{2}{25}\ln 2)T} \quad \boxed{\ln e^{f(x)} = f(x)}$$

$$\ln 2 = \left(\tfrac{2}{25}\ln 2\right)T$$

$$T = \frac{25\ln 2}{2\ln 2}$$

$$T = \frac{25}{2}$$

$$\boxed{T = 12.5 \text{ hours}}$$

9.

$$f(x) = \frac{50x^2 + 38x + 9}{(5x+2)^2(1-2x)} \qquad x \neq -\frac{2}{5} \quad x \neq \frac{1}{2}$$

Given that f(x) can be expressed in the form

$$\frac{A}{5x+2} + \frac{B}{(5x+2)^2} + \frac{C}{1-2x}$$

where A, B and C are constants

(a) (i) find the value of B and the value of C

(ii) show that $A = 0$

(4)

(b) (i) Use binomial expansions to show that, in ascending powers of x

$$f(x) = p + qx + rx^2 + \ldots$$

where p, q and r are simplified fractions to be found.

(ii) Find the range of values of x for which this expansion is valid.

(7)

a) $f(x) = \frac{50x^2+38x+9}{(5x+2)^2(1-2x)} = \frac{A}{5x+2} + \frac{B}{(5x+2)^2} + \frac{C}{1-2x}$

multiply all terms by $(5x+2)^2(1-2x)$

$50x^2 + 38x + 9 = A(5x+2)(1-2x) + B(1-2x) + C(5x+2)^2$

solve for A, B and C by subbing in different values of x

@$x = \frac{1}{2}$ $\quad 50(\frac{1}{2})^2 + 38(\frac{1}{2}) + 9 = A(\frac{9}{2})(0) + B(0) + C(\frac{9}{2})^2$

$(1-2x) = 0$ $\quad 40.5 = \frac{81}{4}C$

so A and B are eliminated $\quad \boxed{C = 2}$

@$x = -\frac{2}{5}$ $\quad 50(-\frac{2}{5})^2 + 38(-\frac{2}{5}) + 9 = A(0)(\frac{9}{5}) + B(\frac{9}{5}) + C(0)^2$

$(5x+2) = 0$ $\quad \frac{9}{5} = \frac{9}{5}B$

$\boxed{B = 1}$

now we know $C = 2$ and $B = 1$ we can sub in any value of x to find A

@$x = 0$ $\quad 50(0)^2 + 38(0) + 9 = A(2)(1) + 1(1) + 2(2)^2$

$9 = 2A + 9$

$2A = 0 \quad \boxed{A = 0}$

Question 9 continued

b) $f(x) = \dfrac{1}{(5x+2)^2} + \dfrac{2}{1-2x}$

$f(x) = (5x+2)^{-2} + 2(1-2x)^{-1}$

first expand $(5x+2)^{-2}$ → use formula for binomial expansion in formula book p5

$2^{-2}\left(1+\frac{5}{2}x\right)^{-2}$

$= \frac{1}{4}\left(1+\frac{5}{2}x\right)^{-2}$

now expand $\left(1+\frac{5}{2}x\right)^{-2}$

$\left(1+\frac{5}{2}x\right)^{-2} = 1 + (-2)\left(\frac{5}{2}x\right) + \dfrac{(-2)(-3)\left(\frac{5}{2}x\right)^2}{2}$

$= 1 - 5x + \frac{75}{4}x^2$

$(5x+2)^{-2} = \frac{1}{4}\left(1+\frac{5}{2}x\right)^{-2} = \frac{1}{4}\left(1 - 5x + \frac{75}{4}x^2\right)$

$= \frac{1}{4} - \frac{5}{4}x + \frac{75}{16}x^2$

then expand $2(1-2x)^{-1}$

$(1-2x)^{-1} = 1 + (-1)(-2x) + \dfrac{(-1)(-2)(-2x)^2}{2}$

$= 1 + 2x + 4x^2$

so $2(1-2x)^{-1} = 2 + 4x + 8x^2$

add $(5x+2)^{-2}$ and $2(1-2x)^{-1}$ to get the final binomial expansion

$f(x) = \frac{1}{4} - \frac{5}{4}x + \frac{75}{16}x^2 + 2 + 4x + 8x^2$

$\boxed{f(x) = \frac{9}{4} + \frac{11}{4}x + \frac{203}{16}x^2}$

to find range

In formula booklet: $|x| < 1$

$\left|\frac{5}{2}x\right| < 1$

ii) $\boxed{|x| < \frac{2}{5}}$

10. **In this question you should show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Given that $1 + \cos 2\theta + \sin 2\theta \neq 0$ prove that

$$\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} \equiv \tan\theta$$

(4)

(b) Hence solve, for $0 < x < 180°$

$$\frac{1 - \cos 4x + \sin 4x}{1 + \cos 4x + \sin 4x} = 3\sin 2x$$

giving your answers to one decimal place where appropriate.

(4)

a) To get $\tan\theta$, we want the numerator to be $\sin\theta$ and the denominator to be $\cos\theta$.

First, find all terms in terms of θ, not 2θ

$$\boxed{\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}}$$

$$\tan\theta = \frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta}$$

$$= \frac{1 - (1 - 2\sin^2\theta) + 2\sin\theta\cos\theta}{1 + (2\cos^2\theta - 1) + 2\sin\theta\cos\theta}$$

use the form of $\cos 2\theta$ which contains $\sin\theta$

form which contains $\cos\theta$

$$= \frac{2\sin^2\theta + 2\sin\theta\cos\theta}{2\cos^2\theta + 2\sin\theta\cos\theta}$$

now factorise sin on the numerator and cos on the denominator, as this would simplify to $\tan\theta$.

$$= \frac{2\sin\theta(\sin\theta + \cos\theta)}{2\cos\theta(\cos\theta + \sin\theta)}$$

simplify

$$= \frac{2\sin\theta}{2\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta$$

$$\boxed{\frac{\sin\theta}{\cos\theta} = \tan\theta}$$

b) If $\theta = 2x$, the two equations are the same

$\tan\theta = 3\sin 2x$

if $\theta = 2x$ $\quad \tan 2x = 3\sin 2x$

Question 10 continued

write tan2x in terms of sin and cos

$$\frac{\sin 2x}{\cos 2x} = 3\sin 2x$$

we cannot divide by sin2x, because it may equal 0, and you cannot divide by 0. Instead try to put all terms on one side.

$$\sin 2x = 3\sin 2x \cos 2x$$

$$\sin 2x - 3\sin 2x\cos 2x = 0$$

$$\sin 2x(1 - 3\cos 2x) = 0$$

we can factor out sin2x instead

$\sin 2x = 0$

so $\cancel{x = 0}$, $90°$, $\cancel{180}$

range is $0 < x < 180$ so only $90°$ is a valid solution.

$1 - 3\cos 2x = 0$

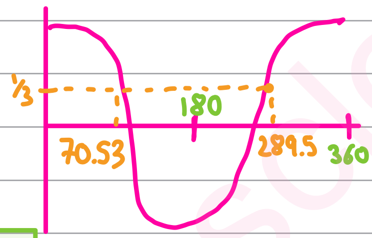
$\frac{1}{3} = \cos 2x$

$2x = 70.53°, 289.5°$

$x = 35.3°, 144.7°$

even though the range is $0 < x < 180$, find solutions for $2x$ because to find x, you need to divide by 2

ie $0 < 2x < 360$



$$\boxed{x = 35.3°, 90°, 144.7°}$$

11.

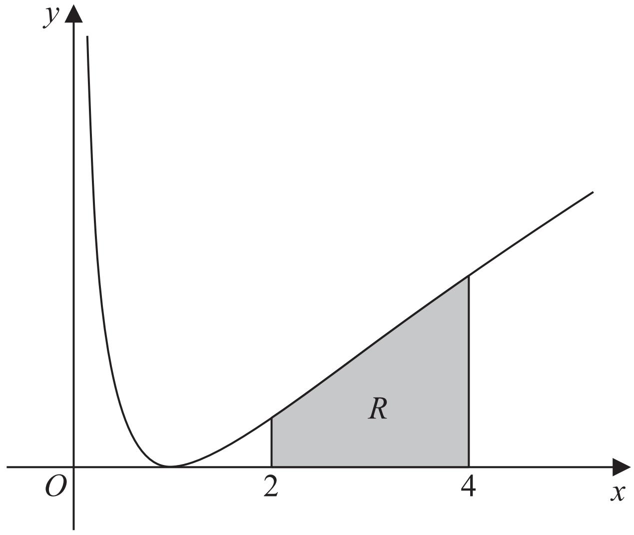


Figure 2

Figure 2 shows a sketch of part of the curve with equation

$$y = (\ln x)^2 \qquad x > 0$$

The finite region R, shown shaded in Figure 2, is bounded by the curve, the line with equation $x = 2$, the x-axis and the line with equation $x = 4$

The table below shows corresponding values of x and y, with the values of y given to 4 decimal places.

x	2	2.5	3	3.5	4
y	0.4805	0.8396	1.2069	1.5694	1.9218

$\nearrow y_0$ $\nwarrow y_n$

(a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of R, giving your answer to 3 significant figures.

(3)

(b) Use algebraic integration to find the exact area of R, giving your answer in the form

$$y = a(\ln 2)^2 + b\ln 2 + c$$

where a, b and c are integers to be found.

(5)

a) Use the formula in the formula booklet, and sub in the values given in the table

$$\boxed{\int_a^b y\,dx \approx \tfrac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\} \qquad h = \frac{b-a}{n}}$$

First find h: $h = \dfrac{4-2}{4} = \tfrac{1}{2}$ (a and b are the limits, and n is the number of trapeziums)

Question 11 continued

now sub in the rest of the values given

$$A = \tfrac{1}{2} \times \tfrac{1}{2}\left(0.4805 + 1.9218 + 2(0.8396 + 1.2069 + 1.5694)\right)$$

$$= \tfrac{1}{4}\left(2.4023 + 2(3.6159)\right)$$

$$= 2.408525$$

$$\boxed{A = 2.41 \ (3sf)}$$

b) $\int (\ln x)^2 \, dx$

we cannot use a substitution to integrate, so use integration by parts instead.

$$\boxed{\int uv' \, dx = uv - \int vu' \, dx}$$

$u = \ln x \quad u' = 1/x$

$v' = \ln x \quad v = ?$

we cannot split $(\ln x)^2$ up into $\ln x$ and $\ln x$, because it is hard to integrate $\ln x$

$u = (\ln x)^2 \quad u' = \frac{2\ln x}{x}$

$v' = 1 \quad v = x$

so, imagine there is a 1 infront of $(\ln x)^2$, and integrate by parts using the 1 and $(\ln x)^2$

use chain rule to find u': $2\ln x \times \frac{1}{x}$

$= \frac{2\ln x}{x}$

$$\int 1(\ln x)^2 dx = x(\ln x)^2 - \int \frac{2x\ln x}{x}$$

$$= x(\ln x)^2 - \int 2\ln x$$

$$= x(\ln x)^2 - 2\int \ln x$$

here, it is difficult to integrate so repeat integration by parts, using 1 and $\ln x$

$u = \ln x \quad u' = 1/x$

$v' = 1 \quad v = x$

$$= x(\ln x)^2 - 2\left(x\ln x - \int 1 \, dx\right)$$

$\frac{x}{x}$

integral of 1 is x

$$= x(\ln x)^2 - 2(x\ln x - x)$$

$$= x(\ln x)^2 - 2x\ln x + 2x$$

now sub in the limits, 4 and 2

$$= \left[x(\ln x)^2 - 2x\ln x + 2x\right]_2^4$$

$$= \left(4(\ln 4)^2 - 8\ln 4 + 8\right) - \left(2(\ln 2)^2 - 4\ln 2 + 4\right)$$

Question 11 continued

$$\boxed{\ln a^b = b\ln a}$$

$\ln 4 = \ln 2^2 = 2\ln 2$ $\quad = 4(\ln 4)^2 - 8\ln 4 + 8 - 2(\ln 2)^2 + 4\ln 2 - 4$

$4(\ln 4)^2 = 4(2\ln 2)^2 = 16(\ln 2)^2$

$8\ln 4 = 8\ln 2^2 = 16\ln 2$ $\quad = 4(2\ln 2)^2 - 16\ln 2 + 8 - 2(\ln 2)^2 + 4\ln 2 - 4$

simplify: $\quad = 16(\ln 2)^2 - 16\ln 2 + 8 - 2(\ln 2)^2 + 4\ln 2 - 4$

$$\boxed{= 14(\ln 2)^2 - 12\ln 2 + 4}$$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

12.

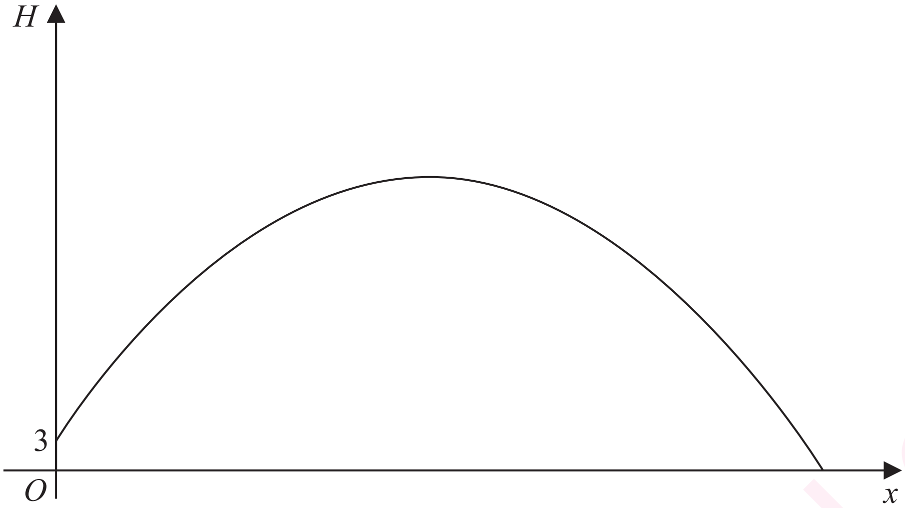


Figure 3

Figure 3 is a graph of the trajectory of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, H metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

Given that the ball

- is hit from a point on the top of a platform of vertical height 3 m above the ground
- reaches its maximum vertical height after travelling a horizontal distance of 90 m
- is at a vertical height of 27 m above the ground after travelling a horizontal distance of 120 m

Given also that H is modelled as a **quadratic** function in x

(a) find H in terms of x

(5)

(b) Hence find, according to the model,

(i) the maximum vertical height of the ball above the ground,

(ii) the horizontal distance travelled by the ball, from when it was hit to when it first hits the ground, giving your answer to the nearest metre.

(3)

(c) The possible effects of wind or air resistance are two limitations of the model. Give one other limitation of this model.

(1)

a) H is a quadratic, so it is in the form ax^2+bx+c

from the graph, we can see that the y intercept : 3

so $C = 3$

P 6 8 7 3 1 A 0 3 8 5 2

Question 12 continued

we also know a point on the graph (27, 120). Sub this in to get an equation in a and b

$$27 = a(120)^2 + b(120) + 3$$
$$24 = 14{,}400a + 120b$$

there are no more points to sub in, so we can differentiate H with respect to x, because we know at the ball's highest point $\frac{dH}{dx} = 0$

$x = 90$ at largest H

$$\frac{dH}{dx} = 2ax + b$$

at max H $x = 90$ $\quad dH/dx = 180a + b = 0$

now we have two equations in a and b which we can solve simultaneously to find a and b

① $180a + b = 0$
② $14{,}400a + 120b = 24$

$120 \times ①$ $\quad 21{,}600a + 120b = 0$

$120(1) - ②$ $= 7{,}200a = -24$

$a = -\frac{1}{300}$

Sub in $a = -\frac{1}{300}$ into (1) to find b

$$-\frac{180}{300} + b = 0$$
$$\frac{180}{300} = b$$
$$b = \frac{180}{300} = 0.6$$

now put a, b, c back into the equation $ax^2 + bx + c = 0$ to find H in terms of x

$$\boxed{H = -\frac{1}{300}x^2 + \frac{3}{5}x + 3}$$

b) the maximum height occurs when $x = 90$ (given in question)

$$H = -\frac{1}{300}(90^2) + \frac{3}{5} \times 90 + 3$$
$$= -27 + 54 + 3$$
$$\boxed{H = 30}$$

Question 12 continued

ii) We need to find x when $H = 0$ (when the ball hits the ground)

$$0 = -\frac{1}{300}x^2 + \frac{3}{5}x + 3$$

multiply by -300:

$$0 = x^2 - 180x - 900$$

solve for x:

$$x = \frac{180 \pm \sqrt{(-180)^2 - (4 \times 1 \times -900)}}{2}$$

$$= \frac{180 \pm \sqrt{32400 + 3600}}{2}$$

$$= \frac{180 \pm 60\sqrt{10}}{2}$$

$$= 90 \pm 30\sqrt{10}$$

$$x = \cancel{-4.868}, 184.87$$

x must be positive

$$\boxed{x = 185\text{ m}}$$

c) The ball is unlikely to travel in a vertical plane

13. A curve C has parametric equations

$$x = \frac{t^2 + 5}{t^2 + 1} \qquad y = \frac{4t}{t^2 + 1} \qquad t \in \mathbb{R}$$

Show that all points on C satisfy

$$(x - 3)^2 + y^2 = 4$$

(3)

Sub the values of x and y in terms of t into $(x-3)^2 + y^2 = 4$ to get an equation in terms of t only.

$$(x-3)^2 + y^2 = \left(\frac{t^2+5}{t^2+1} - 3\right)^2 + \left(\frac{4t}{t^2+1}\right)^2$$

make the denominator of everything t^2+1

$$\boxed{\frac{t^2+5-3(t^2+1)}{t^2+1} \quad \frac{t^2+5-3t^2-3}{t^2+1}}$$

$$= \left(\frac{2-2t^2}{t^2+1}\right)^2 + \left(\frac{4t}{t^2+1}\right)^2$$

$$= \frac{4-8t^2+4t^4}{(t^2+1)^2} + \frac{16t^2}{(t^2+1)^2}$$

if the denominators are the same, you can combine all terms into 1

$$= \frac{4-8t^2+4t^4+16t^2}{(t^2+1)^2}$$

$$= \frac{4t^4+8t^2+4}{(t^2+1)^2}$$

we know we are aiming for $4(t^2+1)^2$ on the numerator, to get 4 as the end value

$$= \frac{4(t^4+2t^2+1)}{(t^2+1)^2}$$

imagine $t^2 = x$

(x^2+2x+1)

$= (x+1)(x+1)$

$= (t^2+1)(t^2+1)$

$$= \frac{4(t^2+1)(t^2+1)}{(t^2+1)^2}$$

$$\boxed{(x-3)^2 + y^2 = 4}$$

14. Given that

$$y = \frac{x-4}{2+\sqrt{x}} \qquad x > 0$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{A\sqrt{x}} \qquad x > 0$$

where A is a constant to be found.

(4)

14. Because we have a fraction, use the quotient rule to differentiate

$$\boxed{\frac{dy}{dx}\,\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}}$$

in formula booklet

$f(x) = x - 4 \quad f'(x) = 1$

$g(x) = 2 + \sqrt{x}$

$= 2 + x^{1/2} \quad g'(x) = \frac{1}{2}x^{-1/2}$

using the formula:

$$\frac{dy}{dx} = \frac{(1\times(2+\sqrt{x})) - ((x-4)(\frac{1}{2}x^{-1/2}))}{(2+\sqrt{x})^2}$$

$$= \frac{2+\sqrt{x} - (x-4)\frac{1}{2}x^{-1/2}}{(2+\sqrt{x})^2}$$

$-(x-4)\frac{1}{2}x^{-1/2}$

$= -x(\frac{1}{2}x^{-1/2}) + 4(\frac{1}{2}x^{-1/2})$

$= -\frac{1}{2}x^{1/2} + 2x^{-1/2}$

$$= \frac{2+\sqrt{x} - \frac{1}{2}\sqrt{x} + 2x^{-1/2}}{(2+\sqrt{x})^2}$$

$2x^{-1/2} = \frac{2}{\sqrt{x}}$

$$= \frac{2+\sqrt{x} - \frac{1}{2}\sqrt{x} + \frac{2}{\sqrt{x}}}{(2+\sqrt{x})^2}$$

multiply top + bottom by $\sqrt{x}$

$$= \frac{2\sqrt{x} + x - \frac{1}{2}x + 2}{\sqrt{x}(2+\sqrt{x})^2}$$

$$= \frac{2\sqrt{x} + \frac{1}{2}x + 2}{\sqrt{x}(2+\sqrt{x})^2}$$

multiply top and bottom by 2

$$= \frac{4\sqrt{x} + x + 4}{2\sqrt{x}(2+\sqrt{x})^2}$$

we know we need $(2+\sqrt{x})^2$ on the bottom to disappear, so it is worth seeing if we can factorise $4\sqrt{x} + x + 4$ into $(2+\sqrt{x})^2$

$$= \frac{(2+\sqrt{x})(2+\sqrt{x})}{2\sqrt{x}(2+\sqrt{x})^2}$$

$$= \frac{(2+\sqrt{x})^2}{2\sqrt{x}(2+\sqrt{x})^2} = \boxed{\frac{1}{2\sqrt{x}}}$$

P 6 8 7 3 1 A 0 4 4 5 2

15. (i) Use proof by exhaustion to show that for $n \in \mathbb{N}, n \leqslant 4$

$$(n+1)^3 > 3^n$$

(2)

(ii) Given that $m^3 + 5$ is odd, use proof by contradiction to show, using algebra, that m is even.

(4)

i) if we use proof by exhaustion, we have to prove $(n+1)^3 > 3^n$ for every number $n \in \mathbb{N}, n \leqslant 4$

natural numbers are positive integers, i.e $1, 2, 3, 4, 5 \ldots \infty$

↳ so natural numbers $n \leqslant 4$ are $1, 2, 3, 4$

show true for $n = 1, 2, 3, 4$ only

$n=1$	$(1+1)^3 = 8$	$3^1 = 3$	$8 > 3$
$n=2$	$(2+1)^3 = 27$	$3^2 = 9$	$27 > 9$
$n=3$	$(3+1)^3 = 64$	$3^3 = 27$	$64 > 27$
$n=4$	$(4+1)^3 = 125$	$3^4 = 81$	$125 > 81$

so if $n \leqslant 4, n \in \mathbb{N}$ then $(n+1)^3 > 3^n$

show working for each number

conclude stating what is in the question

ii) When doing proof by contradiction, you assume the opposite of what you are trying to prove

let m be odd

when p is any integer, m must be odd

$m = 2p + 1$

$$\begin{aligned} m^3 + 5 &= (2p+1)^3 + 5 \\ &= (4p^2 + 4p + 1)(2p+1) + 5 \\ &= 8p^3 + 8p^2 + 2p + 4p^2 + 4p + 1 + 5 \\ &= 8p^3 + 12p^2 + 6p + 6 \\ &= 2(4p^3 + 6p^2 + 3p + 3) \end{aligned}$$

factorise out 2 to show m^3+5 is even

so $m^3 + 5$ is even as it is a multiple of 2

when m is assumed to be odd, $m^3 + 5$ is even. However m^3 is even, which is a contradiction, so m must be even, not odd.